ANALYTICAL RELATIONSHIPS FOR THE INFLUENCE OF PARTICLE SIZE ON PRECIPITATOR DESIGN

by

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Introduction

The realistic design of many types of particulate collection devices involves consideration of the influence of particle size distribution. For an electrostatic precipitator, this influence is appropriately expressed by the integration of the Deutsch equation for a log-normal particle size distribution. The integral which must be evaluated in such cases depends on several parameters and cannot be evaluated exactly in terms of elementary functions. A numerical tabulation of the integral for all parametric cases of interest, while feasible, poses a considerable numerical interpolation problem and is not very suitable for computerization.

In order to express the integral simply and accurately, asymptotic methods are employed in this paper. These methods produce analytical approximations which are expressed in terms of well known functions and the solution of a certain nonlinear algebraic equation. Results of these calculations are given here in forms suitable for either hand or computer calculations. Comparison with numerical calculations shows that the results of the asymptotic analysis are applicable for virtually all cases of practical interest.

Electrostatic Precipitator Theory

In an electrostatic precipitator, suspended dust particles in a gas are electrically charged and migrate to collecting surfaces where they are captured. A force balance on a single charged particle in a quiescent gas under the influence of an electric field produces the following expression for the particle migration velocity:

$$w = \frac{E_o E_p d_p}{4\pi\mu} \tag{1}$$

The collection efficiency of a precipitator in turbulent flow is then given by the Deutsch equation

$$\eta = 1 - e^{-A_p w/Q} \tag{2}$$

However, the value of w calculated according to first principles from eqn. (1) with its underlying simplifying assumptions may be several times too high because of effects unaccounted for in its derivation. These include multiple particles, uneven gas distribution, particle reentrainment, and high dust resistivity, as comprehensively discussed by White.[11, 12] Therefore, to predict the operation of an actual precipitator, the theoretical migration velocity is replaced by a precipitation rate constant based on experience.[6]

Log-Normal Particle Size Distribution

Moreover, the particles encountered in practice are not uniform in size as we have so far tacitly assumed, but rather are made up of a continuous distribution of sizes. Specifically, most powders of industrial significance are log-normally distributed. The frequency of occurrence of a given diameter particle plotted against the logarithm of that size produces the familiar bell-shaped curve. Alternatively, the logarithm of the particle diameter (d_p) graphs as a straight line against cumulative weight percent less than that size. An example of the latter plot is shown in Figure 1. Two parameters, the mass median diameter ($\bar{d}_p$) and the geometric standard deviation (σ) completely specify the distribution. The mass median diameter is the diameter at the 50% point, and σ is given by either of the following ratios:

$$\bar{d}_p/d_p \text{ at } 15.9\% \text{ or } d_p \text{ at } 84.1\%/\bar{d}_p$$

A more complete explanation of fine particle statistics is contained elsewhere.[5]

Collection Efficiency Log-Normal Distribution

By lumping all non-size related quantities into one parameter

$$k = \frac{E_o E_p A_p}{4\pi\mu Q} \tag{3}$$

eqn. (2) can be written

$$\eta = 1-e^{-kd_p} \tag{4}$$

Eqn. (4) is exponential in character although k, like w, is empirical. This function, shown in Fig. 2, is an example of the so-called grade efficiency curve discussed by Stairmand.[8-10] Typically, larger particles are collected much more readily than smaller ones.

With the grade-efficiency function defined, the collection efficiency for each individual particle size in the distribution is completely determined. The overall efficiency of collection (η) is obtained by summing up these individual contributions

$$\eta = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2} - ae^{bt}} dt \tag{5}$$

where $a \equiv k\bar{d}_p$ and $b \equiv \ln \sigma$

Although attention has been focused on electrostatic precipitators, the technique to be outlined below applies as well to other types of particulate collection devices whose grade-efficiency curves can be approximated

by eqn. (4). Typical grade-efficiencies replotted on semi-logarithmic coordinates from Stairmand's tabulations and curves are shown in Fig. 3 for several such devices. Increasingly efficient collection is reflected by a larger value of k.

Numerical Integration

The integral of eqn. (5) can be computed directly by numerical quadrature, resulting in the curves presented in Fig. 4. For some purposes, these curves will be entirely satisfactory. More than likely, however, collection efficiency will be desired for a set of input parameters a and b not corresponding exactly to any of these curves, requiring either an interpolation or a complete numerical re-evaluation of the integral. In other words, the functional dependence of the efficiency on the input parameters is not shown explicitly. For this reason, simple and accurate analytical approximations for this integral have been developed.

Approximate Representation of $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-t^2}{2} - ae^{bt}} dt \equiv I$

This infinite integral is seen to converge for all values of a and b. The integrand $\to 0$ for both $t \to -\infty$ and $t \to \infty$ and thus, since the integrand is always positive, it must have a maximum value at some point t^*. At the maximum point t^*, the usual condition for an interior maximum requires that

$$t^* + abe^{bt^*} = 0 \tag{6}$$

It is easily verified that there can be but one maximum point t^* for any given set of values of a and b. Thus t^* depends on both a and b. However, by changing the variables to $t^* \equiv a^{1/2} u^*$ and $C \equiv a^{1/2} b$, eqn. (6) is reduced to a form involving the single parameter C as follows:

$$u^* + Ce^{Cu^*} = 0 \tag{7}$$

Since the integrand involves exponentials, it is natural to hope that by investigating the maximum of the integrand and suitably changing variables, the original integral might be transformed to one of the Laplace type, which could then be evaluated approximately for certain limiting values of a or b.[2] However, in this problem a transformation of this type does not yield the desired Laplace integral, but the transformation is nevertheless extremely useful since it does lead ultimately to a resolution of the original problem.

Some details of the analysis involved in approximating the integral are given in the appendix for the reader who is interested. Here we will present the essential results of the analysis. First we note that the approximation formulas depend on the solution of nonlinear eqn. (7). The solution of this equation is indicated graphically in Fig. 5. Approximate analytical solutions of this equation accurate to better than 5% are developed in Appendix I. These approximations may be used directly, or, if more accuracy is desired, they may be used as first guesses for iteration schemes that are also outlined in this appendix. This approach is very attractive for use on the digital computer.

The most useful approximation of the original integral I is as follows:

$$I \doteq \frac{e^{\frac{-A^2b^2}{2}} - A}{(1+Ab^2)^{1/2}} \quad (a \to \infty,\ a \to 0,\ b \to 0) \tag{8}$$

The quantity A in eqn. (8) is related to the solution of nonlinear eqn. (7) since $Ab^2 \equiv -Cu^*$ and the solution of eqn. (7) is expressed in terms of this quantity.

Eqn. (8) is a simple expression which approximates the original integral very well, as shown in Table 1. In Appendix II, it is proved that eqn. (8) is valid in the limit of either $a \to \infty$, $a \to 0$, or $b \to 0$. The fact that eqn. (8) applies for so many limiting cases gives some reason for its excellent agreement with numerical results over such a wide range of a and b values. Except for the impractical case of large b(log-normal probability particle size distribution approaching a vertical line), the approximation would produce results coincident with the curves of Fig. 4.

Eqn. (8) represents the first approximation of an asymptotic expansion for the limiting cases quoted, and it is generally possible to increase the accuracy of the approximation by adding in some correction terms. However, adding only the next uniformly valid contribution for $a \to 0$, $a \to \infty$, $b \to 0$ causes the formula to become very much more complicated. Moreover, the comparison between eqn. (8) and the numerical results shown in Table 1 indicates that correction terms are not really necessary.

Although eqn. (8) seems to represent an appropriate approximation to the original integral I for current needs, it is difficult to anticipate whether some cases that seem unimportant now will prove to be significant at a later time. For this reason and for the sake of completeness, certain other approximations to I are given in Appendix III.

Approximate Representation of $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\frac{-t^2}{2} - ae^{bt}} dt \equiv J$

Frequently an estimate of the particle size distribution of the uncollected material is required in addition to an overall efficiency prediction. This would be important where several collecting devices are to be installed in series. For example, it is not uncommon for cyclones to be installed upstream of a precipitator. In this case, the quantity and distribution of the particulates escaping from the cyclones would be used as the feed to the precipitator, and the calculation repeated.

To calculate the particle size distribution, it is desirable to approximate the above integral where $z = \ln(dp/\overline{dp})/b$ is now a parameter in addition to a and b. It appears possible to do this in a manner analogous to that for the previous case ($z \to \infty$). Again, the details are indicated in the appendix (Appendix IV). The result is

$$(z<t^*) \qquad J \doteq \frac{e^{\frac{-z^2}{2} - E + \frac{(z+Eb)^2}{2(1+b^2E)}}}{\pi^{1/2}(1+b^2E)^{1/2}} \int_{s = -\frac{z+bE}{2^{1/2}(1+b^2E)^{1/2}}}^{\infty} e^{-s^2} ds \qquad (9)$$

where $E \equiv a\, e^{+bz}$.

Eqn. (9) holds only for $z<t^*$. For $z>t^*$, we have instead

$$(z>t^*) \qquad J \doteq I - \frac{e^{\frac{-z^2}{2} - E + \frac{(z+bE)^2}{2(1+b^2E)}}}{\pi^{1/2}(1+b^2E)^{1/2}} \int_{s = \frac{z+bE}{2^{1/2}(1+b^2E)^{1/2}}}^{\infty} e^{-s^2} ds \qquad (10)$$

The approximation given for J in eqns. (9) and (10) involve the well known error function which is widely tabulated and for which there are simple computer approximations.[1] Formulas (9) and (10) are shown in example 1 following to approximate well to the integral J.

Typical Applications of the Results

The above functions provide an analytical tool to evaluate the effect of inlet particule size distribution on collection efficiency at constant precipitator flow and field conditions. This treatment is intermediate between the simplified Deutsch equation based on a uniform particle size, eqn. (2), and a detailed systems analysis.[6] The use of these asymptotic relationships will be demonstrated in the numerical examples given below.

Example 1

Problem:

Compute the overall efficiency of the precipitator whose grade-efficiency curve is given in Fig. 2 operating on the inlet dust distribution shown in Fig. 1. Evaluate the particle size distribution of the uncollected material.

Solution:

By replotting Fig. 2 on semi-logarithmic coordinates as in Fig. 3 and determining the slope, one finds k = 0.46 reciprocal microns. From Fig. 1, $\bar{d}_p$ = 12 microns and σ = 2.8. Therefore, $a = k\bar{d}p = 5.52$, $b = \ln \sigma = 1.03$, and $C = a^{1/2}b = 2.42$. From Fig. 5, $Ab^2 = 1.42$ giving A = 1.34. From eqn. (8),

$$\eta = 1 - \frac{e^{\frac{-A^2b^2-A}{2}}}{(1+Ab^2)^{1/2}} = 1 - 0.0654 \doteq 93.4\%$$

(rounded down)

A refinement of these calculations including a computer solution of eqn. (7) gives an efficiency of 1-0.065436 compared to the above figure. An efficiency of 1-0.065370 is obtained by direct numerical integration of eqn. (5). Agreement of these results is excellent.

The particle size distribution in terms of the fraction lost below a given size can be found by considering the integral J with upper limit given by $z = \ln (d_p/\bar{d}_p)/b$. J can be approximated by eqns. (9) or (10) depending upon the value of z. To obtain the cumulative particle size distribution, J must be divided by I, the infinite integral. The results of these calculations, set forth in Table 2, show quite good agreement for this example.

In addition, the preceding calculations can be compared with a more commonly employed method. The inlet dust distribution is broken up into a finite number of narrow size ranges, the amount collected for each range is determined, and these values are summed to find the overall collection efficiency. An example of this procedure is given by Stairmand.[8, 10] The calculations are summarized in Table 3. The overall collection efficiency is again 93.4%. The distribution of escaping material is slightly finer than the distribution obtained by direct numerical integration of the function J, shown previously in Table 2. The "exact" numerical integration distribution, with $\bar{d}_p$ of 2.7 microns and σ of 1.9, lies between this approximate distribution and the one obtained using the asymptotic expressions.

For this illustrative calculation, entries in Table 3 corresponding to the inlet particle size and collection efficiency have been computed from the appropriate functional forms rather than graphically as might normally be done for a routine calculation. The differences in efficiency and distribution are therefore caused only from taking a series of finite intervals in which the collection efficiency is assumed to be constant at its midpoint value.

In this example, the asymptotic analytical expressions derived in this paper have been shown to produce an overall efficiency equivalent to that obtained from direct numerical integration and a commonly employed approximate technique. In addition, the particle size distributions of all three methods are roughly comparable, with differences of the order of a few percent. The asymptotic functions are especially attractive for hand calculations when only the overall efficiency is desired. When a complete particle size distribution is required, the calculations involving eqns. (9) and (10) are more tedious than the method outlined by Stairmand, which would probably be used. When computer facilities are available, the asymptotic functions represent a savings in computer time over a standard numerical quadrature and are expected to require about the same order of magnitude of time as that used by the Stairmand type of calculation.

Example 2

Problem:

To increase collection efficiency, the precipitator of example 1 is to be doubled in size by increasing the length of its plates in the direction of gas flow. Calculate the new efficiency accounting for particle size distribution and by using the simplified Deutsch equation.

Solution:

To account for a change in plate length, one must vary k which is directly proportional to this dimension. In example 1, the parameters of the inlet dust and k = 0.46 led to an efficiency of 93.4%. When the size is doubled, k becomes 0.96 and the computation proceeds in a similar manner, giving an efficiency of 97.9%.

This calculation can be compared to an entirely different technique. As is sometimes done, eqn. (2), the simplified Deutsch equation for a monodisperse particulate, can be employed and thereby circumvent the use of an integral and all the difficulties involved in its evaluation. In this procedure, the observed 93.4% efficiency can be used to fix the value of the argument of the exponential at 2.7. Doubling the plate length makes this parameter 5.4 and gives an efficiency of 99.5%. Although the two efficiencies, 97.9% and 99.5%, appear close, the loss rates differ significantly by a factor of 4.

The efficiency as a function of length is plotted in Fig. 6. The asymptotic expression gives the efficiency shown by the solid curve, while the simplified Deutsch equation produces the broken line. The two provide equivalent results only in the vicinity of 93.4% efficiency where the Deutsch equation exponent was fitted.

Thus, the Deutsch equation predicts a greater efficiency at a given increment in length or a shorter additional length required to achieve a given increase in efficiency. This is the same sort of behavior demonstrated

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TABLE 1

COMPARISON OF APPROXIMATE METHOD WITH NUMERICAL INTEGRATION

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-t^{1/2}}{2} - ae^{bt}} dt$$

a	σ=1, b=ln σ = 0		σ=1.1052, b=0.1		σ=2, b=0.6931		σ=3, b=1.0986		σ=10, b=2.3026		σ=22026, b=10	
	Approx.	Num. Int.	Approx.	Num. Int.	Approx.	Num. Int.	Approx.	Num. Int.	Approx.	Num. Int.	Approx.	Num. Int.
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	0.3679	0.3679	0.3679	0.3679	0.3757	0.3733	0.3892	0.3848	0.4230	0.4196	0.4359	0.4773
2	0.1353	0.1353	0.1367	0.1367	0.1847	0.1838	0.2278	0.2263	0.3173	0.3185	0.4009	0.4500
3	0.04979	0.04979	0.05126	0.05127	0.1036	0.1031	0.1526	0.1520	0.2618	0.2643	0.3818	0.4341
4	0.01832	0.01832	0.01941	0.01941	0.06302	0.06279	0.1099	0.1097	0.2259	0.2288	0.3690	0.4229
5	0.006738	0.006738	0.007416	0.007415	0.04061	0.04049	0.08301	0.08293	0.2001	0.2032	0.3593	0.4143
6	0.002479	0.002479	0.002859	0.002858	0.02732	0.02726	0.06484	0.06484	0.1804	0.1835	0.3515	0.4072
7	0.0009119	0.0009118	0.001111	0.001111	0.01902	0.01898	0.05195	0.05199	0.1648	0.1678	0.3451	0.4013
8	0.0003355	0.0003354	0.0004356	0.0004355	0.01361	0.01359	0.04247	0.04253	0.1520	0.1549	0.3397	0.3962
9	0.0001234	0.0001234	0.0001721	0.0001721	0.009962	0.009951	0.03528	0.03535	0.1412	0.1441	0.3349	0.3917
10	0.00004540	0.00004540	0.00006854	0.00006854	0.007436	0.007429	0.02971	0.02977	0.1321	0.1349	0.3307	0.3877
11	0.00001670	0.00001670	0.00002753	0.00002751	0.005645	0.005640	0.02530	0.02536	0.1241	0.1269	0.3269	0.3841
12	0.000006144	0.000006144	0.00001113	0.00001113	0.004347	0.004345	0.02176	0.02182	0.1172	0.1198	0.3236	0.3808
13	0.000002260	0.000002260	0.000004536	0.000004534	0.003391	0.003390	0.01887	0.01893	0.1111	0.1136	0.3205	0.3778
14	0.0000008315	0.0000008315	0.000001862	0.000001861	0.002676	0.002675	0.01649	0.01654	0.1056	0.1081	0.3176	0.3750

TABLE 2

COMPARISON OF APPROXIMATE AND NUMERICALLY INTEGRATED UNCOLLECTED PARTICLE SIZE DISTRIBUTIONS FOR EXAMPLE 1

d_p(microns)	$J_{(approx.)}$*	Cum. Wt.%	$J_{(numerical)}$	Cum. Wt.%
1	0.00557	8.5	0.00563	8.6
2	0.0209	32.0	0.0219	33.5
3	0.0347	53.1	0.0372	56.9
4	0.0471	72.0	0.0481	73.6
5	0.0537	82.1	0.0550	84.1
6	0.0579	88.5	0.0592	90.5
7	0.0607	92.8	0.0618	94.5
8	0.0624	95.4	0.0632	96.6
9	0.0635	97.1	0.0641	98.0
10	0.0642	98.2	0.0646	98.8
∞	0.065436	100.	0.065370	100.

*In this example, $d_p \cong 2.96$ corresponds to $z=t^*$ and is therefore where we switch from eqn. (9) to eqn. (10).

TABLE 3

CALCULATION OF OVERALL COLLECTION EFFICIENCY FOR EXAMPLE 1 USING METHOD OUTLINED BY STAIRMAND

Size of Grade (microns)	% in Grade at Inlet	Collection Efficiency at Mean Size (%)	Overall Collection (%)	Cum. Wt.% Collected	Cum. Wt.% Escaping
0-1	0.8	20.6	0.165	0.18	9.7
1-2	3.3	49.8	1.64	1.9	34.9
2-3	4.7	68.3	3.21	5.4	57.7
3-4	5.4	80.0	4.32	10.0	74.1
4-5	5.6	87.4	4.89	15.2	84.9
5-6	5.2	92.4	4.80	20.4	91.0
6-7	5.0	95.0	4.75	25.5	94.9
7-8	4.7	96.8	4.55	30.3	97.1
8-9	4.3	98.0	4.21	34.8	98.5
9-10	4.0	98.7	3.95	39.1	99.3
10-15	15.6	99.7	15.55	55.7	100
15-20	10.5	100	10.5	67.0	100
20+	30.9	100	30.9	100	100
	100		93.435		

Figure 1

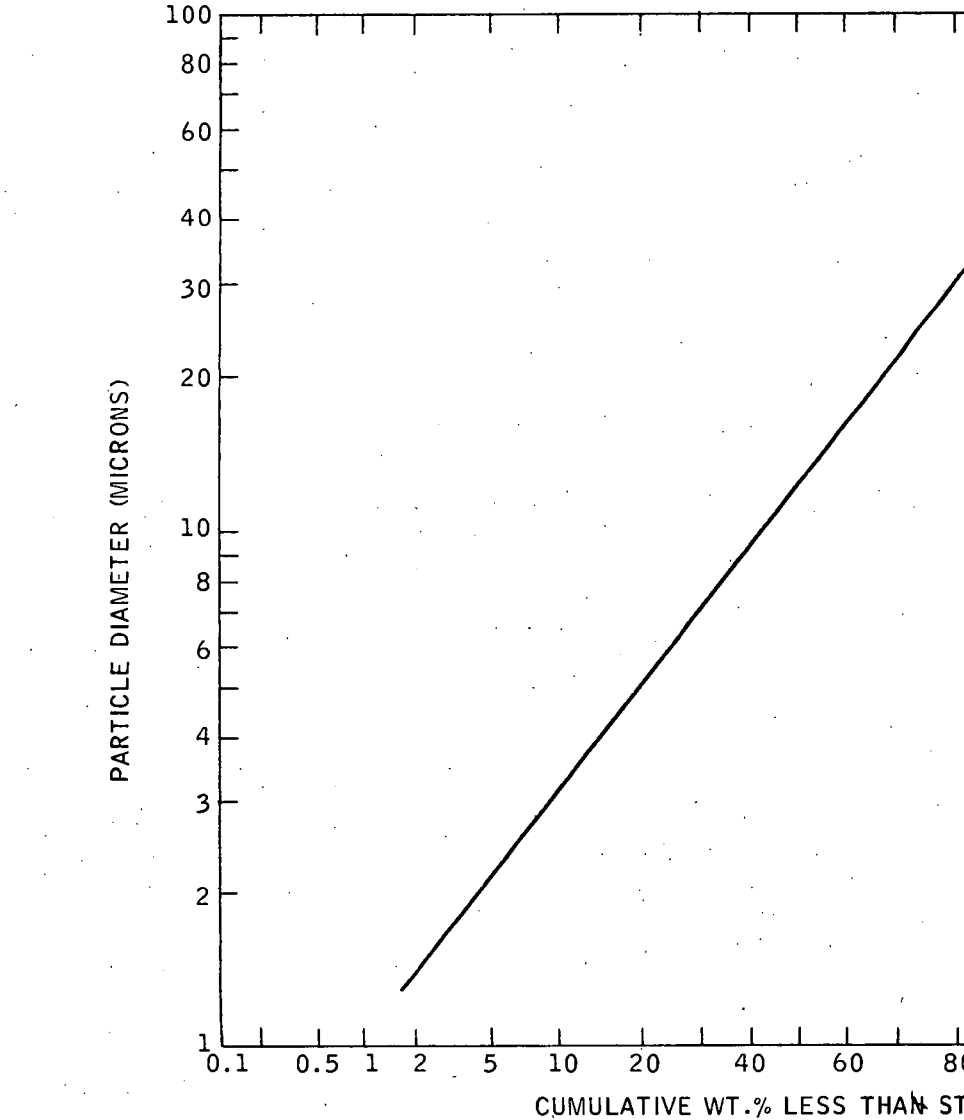

PARTICLE DIAMETER (MICRONS)
100
80
60
40
30
20
10
8
6
4
3
2
1
0.1
0.5
1
2
5
10
20
40
60
80
90
95
98
99.5
99.9
CUMULATIVE WT.% LESS THAN STATED SIZE

Figure 2

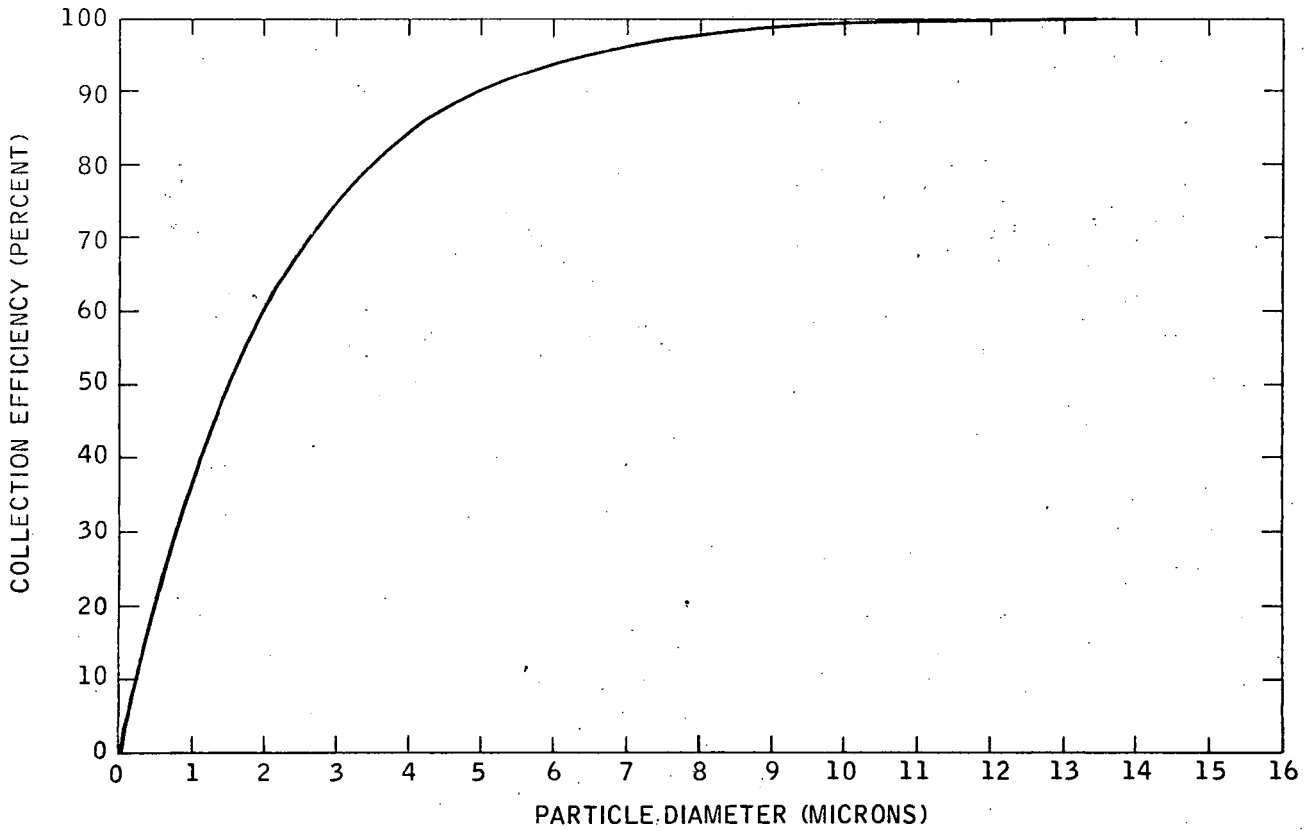

COLLECTION EFFICIENCY (PERCENT)
100
90
80
70
60
50
40
30
20
10
0
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
PARTICLE DIAMETER (MICRONS)

Figure 3

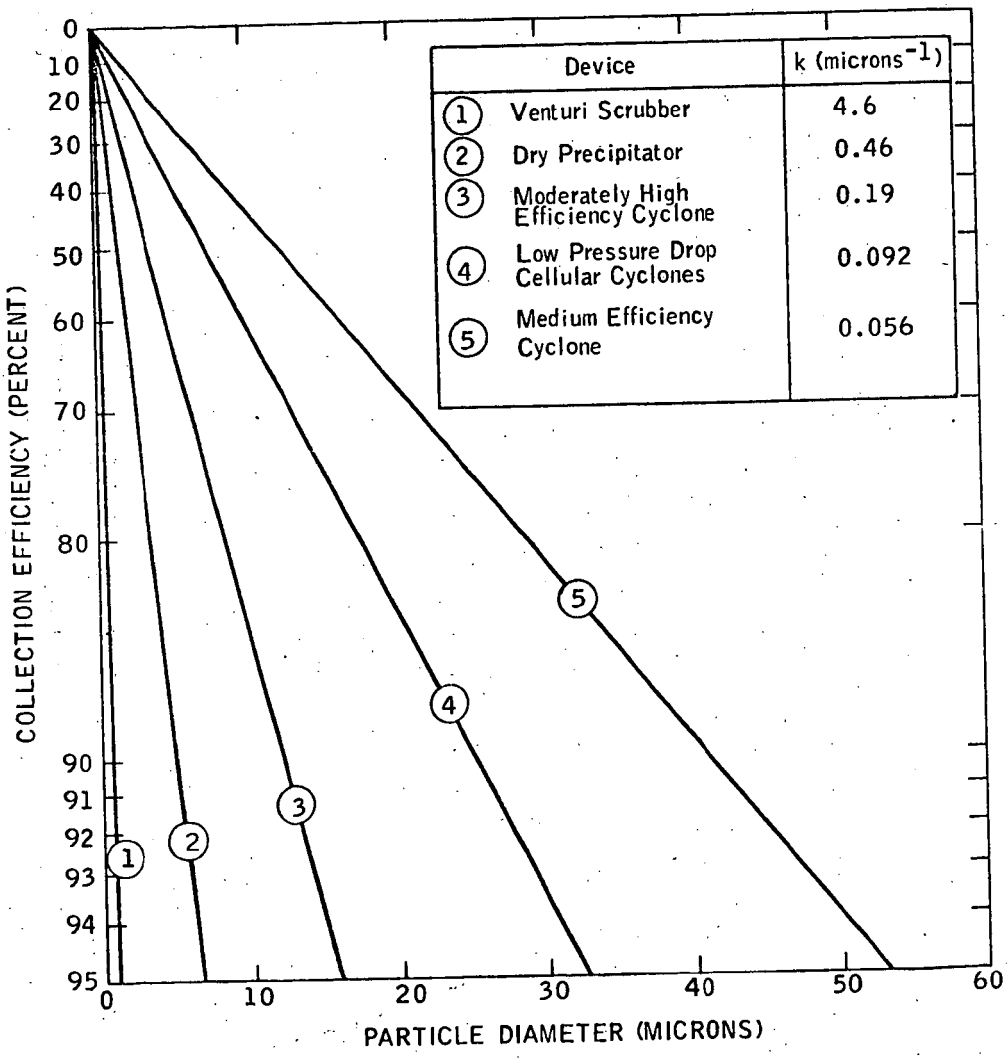

Device
k (microns^{-1})
1 Venturi Scrubber 4.6
2 Dry Precipitator 0.46
3 Moderately High Efficiency Cyclone 0.19
4 Low Pressure Drop Cellular Cyclones 0.092
5 Medium Efficiency Cyclone 0.056
COLLECTION EFFICIENCY (PERCENT)
0
10
20
30
40
50
60
70
80
90
91
92
93
94
95
PARTICLE DIAMETER (MICRONS)
0
10
20
30
40
50
60

Figure 4

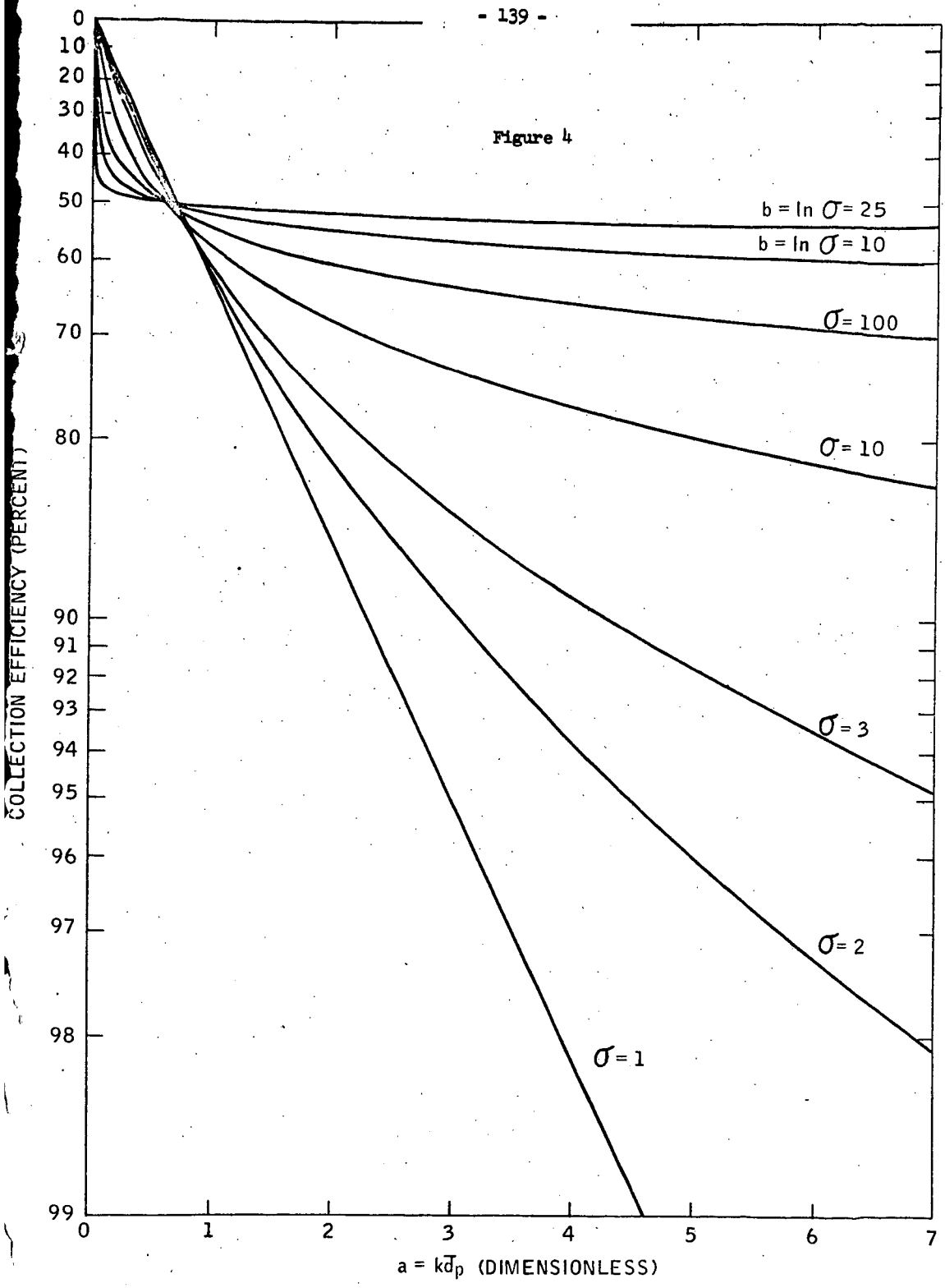

COLLECTION EFFICIENCY (PERCENT)
0
10
20
30
40
50
60
70
80
90
91
92
93
94
95
96
97
98
99
b = ln σ = 25
b = ln σ = 10
σ = 100
σ = 10
σ = 3
σ = 2
σ = 1
0
1
2
3
4
5
6
7
a = kd̄p (DIMENSIONLESS)

Figure 5

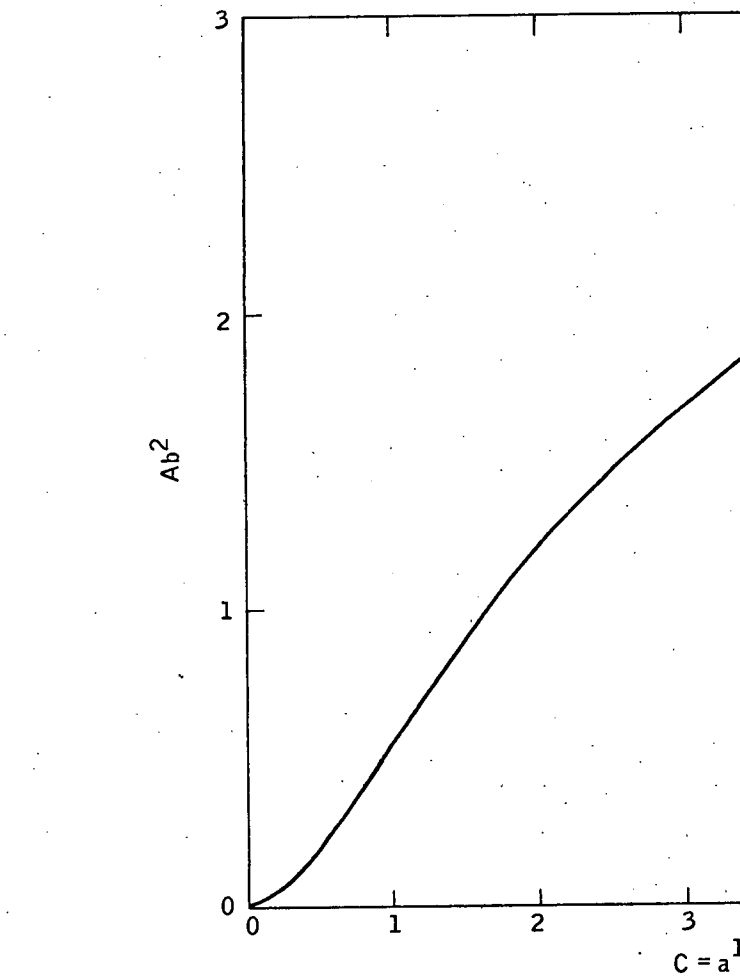
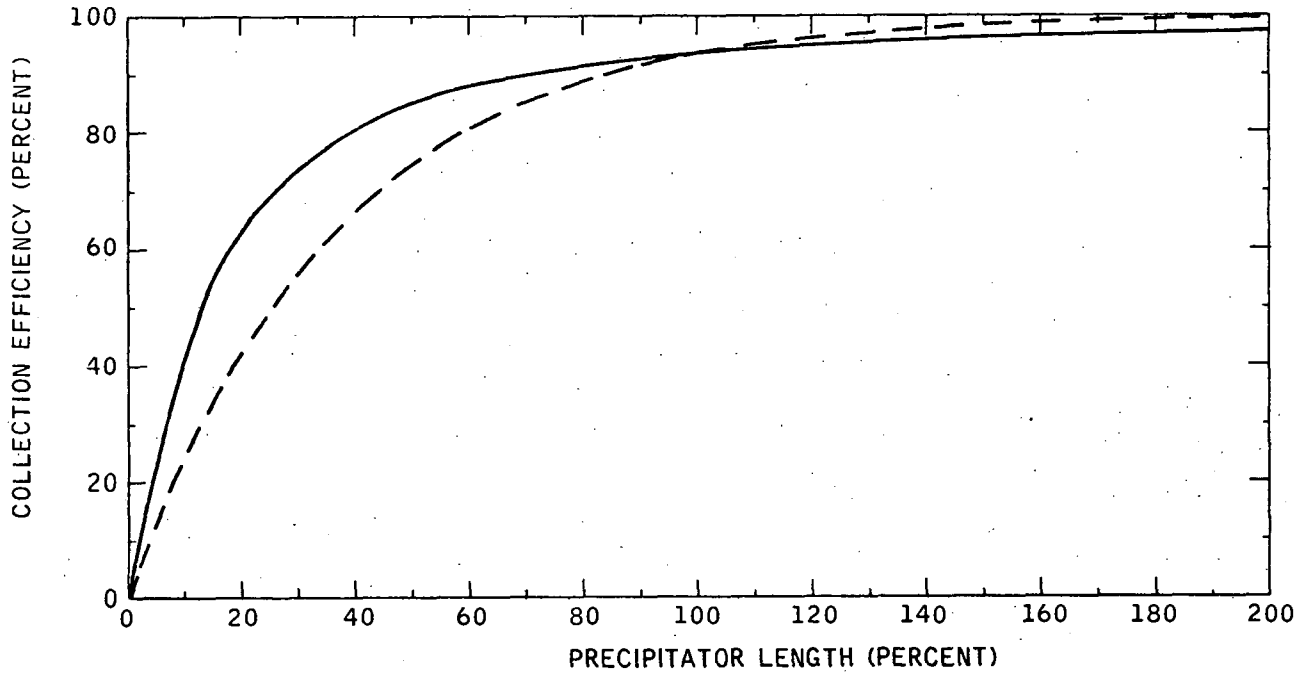


Figure 6


COLLECTION EFFICIENCY (PERCENT)
100
80
60
40
20
0
0
20
40
60
80
100
120
140
160
180
200
PRECIPITATOR LENGTH (PERCENT)

Notation

$a = k\bar{d}_p$

$A = ae^{bt*} = -Cu*/b^2 = -t*/b$

A_p = area of collection plates

$b = \ln \sigma$

$C = a^{1/2}b$

d_p = particle diameter

$\bar{d}_p$ = mass median diameter

e = base of natural logarithms

$E = ae^{bz}$

E_o = charging electrical field intensity

E_p = collection electrical field intensity

I = infinite integral in eqn. (5)

J = integral I with variable upper limit

k = collection parameter

Q = gas volumetric flow

s = variable of integration

t = variable of integration

t* = location of maximum point

$u* = t*/a^{1/2}$

w = effective migration velocity

z = variable upper limit of integral J = $\ln(dp/\bar{d}_p)/b$

Greek Letters

η = collection efficiency

μ = gas viscosity

π = numerical constant

σ = geometric standard deviation

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Appendix I

The solution of $u^* + Ce^{Cu^*} = 0$.

This equation is easily solved numerically by means of a suitable interation method if appropriate starting values are used. Suitable procedures include the Newton method or the ordinary iteration method accelerated by means of the δ^2 process.[4] The latter method was used to obtain the numerical values shown in Fig. 5 and Table A-1. Asymptotic solutions of the equation were used to yield reasonable starting values. The two asymptotic solutions for $-Cu^* \equiv Ab^2$ are

$$G_1(C) = \frac{2C^2\ (1+C^2)}{2(1+C^2)^2-C^4} \qquad (C \to 0) \qquad \text{(A1-1)}$$

and

$$G_2(C) = 2\ell n\ C - \ell n(2\ell n\ C) \qquad (C \to \infty) \qquad \text{(A1-2)}$$

These two asymptotes are compared with the numerical values in Table A-1. Since the $C \to \infty$ solution, eqn. (A1-2) involves a logarithmic scale, it only becomes accurate at extremely large (and impractical) values of C. However, it was observed that the difference between the numerical values and eqn. (A1-2) is nearly constant for $C \geq 2.5$, and thus in the table values of $G_2(C)+0.3$ are seen to yield accurate results for $C \geq 2.5$.

The asymptotic solutions given above are very useful in the asymptotic analysis of the integral I. Eqns. (A1-1) and (A1-2) show the asymptotic behavior of the quantity A with respect to a and b. For a or $b \to 0$, $A \to a$ and for $a \to \infty$, $A \to 2\ \dfrac{\ell n(a^{1/2}b)}{b^2}$.

TABLE A-1

COMPARISON OF NUMERICAL AND ASYMPTOTIC SOLUTIONS OF EQN. (7)

	Numerical	Asymptotic		
C	$-Cu^* = Ab^2$	$G_1(C)$	$G_2(C)$	$G_2(C) + 0.3$
0	0	0		
0.1	0.0099	0.0099		
0.3	0.0828	0.0828		
0.5	0.2039	0.2040		
1	0.5671	0.5614		
2	1.202	1.176	1.060	1.360
2.5	1.456	1.372	1.227	1.527
3	1.679	1.513	1.410	1.710
4	2.053		1.753	2.053
5	2.360		2.050	2.350
6	2.620		2.307	2.607
8	3.045		2.734	3.034
10	3.386		3.078	3.378
20	4.490		4.201	4.501
1000	11.38		11.19	11.49
10,000	15.67		15.51	

$$G_1(C) = \frac{2C^2(1+C^2)}{2(1+C^2)^2 - C^4}, \quad G_2(C) = 2\ln C - \ln(2\ln C)$$

Appendix II

Evaluation of $\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{\frac{-t^2}{2} - ae^{bt}} dt \equiv I$

We proceed by splitting the integral by means of $\int_{-\infty}^{\infty} = \int_{-\infty}^{t*} + \int_{t*}^{\infty}$ where $t*$ is the solution of eqn. (6). Thus $I = I_1 + I_2$ where $I_1 = \int_{-\infty}^{t*}$ and $I_2 = \int_{t*}^{\infty}$. First consider I_2, which turns out to be much easier to analyze than I_1. (It happens that to the first approximation, I_1 and I_2 are equal). By translating the origin of coordinates to $t*$ through $t-t* = u$, we get

$$I_2 = \frac{e^{\frac{-b^2A^2}{2}}}{(2\pi)^{1/2}} \int_{u=o}^{\infty} e^{\frac{u^2}{2} + Abu - Ae^{bu}} du \qquad \text{(A2-1)}$$

Since the new integrand, by way of its construction, is a maximum near u=0, it is natural to approximate to the integrand in this vicinity in hopes that though the approximation may not be good in other regions, these regions contribute only a small part to the integral. The best procedure seems to be the following. Write

$$e^{bu} = 1 + bu + \frac{b^2u^2}{2} + e^{bu} - 1 - bu - \frac{b^2u^2}{2}.$$

Then expand the exponential in eqn. (A2-1) to get

$$I_1 = \frac{e^{\frac{-b^2A^2}{2} - A}}{(2\pi)^{1/2}} \int_{u=o}^{\infty} e^{-(1+b^2A)\frac{u^2}{2}} \left[1 - A(e^{bu} - 1 - bu - \frac{b^2u^2}{2})..\right] du \qquad \text{(A2-2)}$$

where the error incurred by taking 1 for the quantity in square brackets is negative and less in magnitude than the next term.

By performing the integration we get

$$I_2 = \frac{e^{\frac{-b^2A^2}{2} - A}}{(2\pi)^{1/2}} \left[\frac{\pi^{1/2}}{(2g)^{1/2}} - A\left\{ \left(\frac{2}{g}\right)^{1/2} e^{\frac{b^2}{2g}} \int_{s=\frac{-b}{(2g)^{1/2}}}^{\infty} e^{-s^2} ds - \left(\frac{\pi}{2g}\right)^{1/2} - \frac{b}{g} - \frac{b^2\pi^{1/2}}{(2g)^{3/2}} \right\} \right] \tag{A2-3}$$

where $g \equiv (1 + b^2A)$

The error made by approximating the quantity in square brackets by the first term is negative and less in magnitude than the next term. It is easily shown eqn. (A2-3) that the relative error associated with using just the first term in square brackets goes to zero for either $a \to 0$, $a \to \infty$ or $b \to 0$. The integral I is transformed by means of $t^* - t = s$ and becomes

$$I_1 = \frac{e^{\frac{-b^2A^2}{2} - A}}{(2\pi)^{1/2}} \int_{s=o}^{\infty} e^{\frac{-s^2}{2} - Abs - Ae^{-bs}} ds \tag{A2-4}$$

Here again the integrand is approximated near its maximum (s=0) and to a first approximation for either $a \to 0$, $a \to \infty$ or $b \to 0$

$$I_1 \to \frac{e^{\frac{-b^2A^2}{2} - A}}{2g^{1/2}}$$

However, in the case of I_1, the demonstration of the validity of this result is much more involved than it was in the case of I_2 and the lengthy details are omitted.

Appendix III

Other Asymptotic Expansions to $I = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{\frac{-t^2}{2} - ae^{bt}} dt$

For $b \to \infty$, $I \to 1/2$ in accordance with

$$I = 1/2 + R_1, \quad |R_1| \leq \frac{1}{\sqrt{2\pi}} \max \left[\frac{a}{b}, \frac{e^{-a}}{ab}\right] \tag{A3-1}$$

Note the curve for b=25 of Fig. 4 is approaching this limit.

For $b \to 0$ we have

$$I = e^{\frac{-A^2b^2}{2}} \left[1 - A\left(e^{\frac{b^2}{2}} - 1\right) ..\right] \tag{A3-2}$$

Eqn. (A3-2) is useful since it has the property that the error made by taking the term in square brackets equal to 1 is negative and less in magnitude than $A\left(e^{\frac{b^2}{2}} - 1\right)$.

For $a \to 0$

$$I \sim \sum_{i=0}^{\infty} \frac{(-1)^i a^i e^{\frac{b^2 i^2}{2}}}{i!} = 1 - ae^{\frac{b^2}{2}} + \frac{a^2 e^{2b^2}}{2!} - \frac{a^3 e^{\frac{9b^2}{2}}}{3!} .. \tag{A3-3}$$

This series diverges for any $b > 0$, but it is asymptotic in the sense that the error is of the same sign as the first neglected term and less than it in magnitude. Therefore, the series is of use computationally for sufficiently small values of a.

Appendix IV

Evaluation of $\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{z} e^{\frac{-t^2}{2} - ae^{bt}} dt = J$

It is convenient to consider separately the cases $z < t^*$ and $z > t^*$. For $z < t^*$, the maximum of the integrand is at z and accordingly we move the origin of coordinates to z and approximate to the integrand in this vicinity. Thus J becomes

$$J = \frac{e^{\frac{-z^2}{2}}}{(2\pi)^{1/2}} \int_{\omega=o}^{\infty} e^{z\omega - \frac{\omega^2}{2} - /Ee^{-b\omega}} d\omega \qquad \text{(A4-1)}$$

where $E \equiv ae^{bz}$

By approximating the exponential near $\omega = 0$, we get

$$J \doteq \frac{e^{\frac{-z^2}{2} - E}}{(\pi)^{1/2}(1+b^2E)^{1/2}} e^{\frac{(bE+z)^2}{2(1+b^2E)}} \int_{s=\frac{(bE+z)}{2^{1/2}(1+b^2E)^{1/2}}}^{\infty} e^{-s^2} ds \qquad (z \leq t^*) \quad \text{(A4-2)}$$

The integral appearing in eqn. (A4-2) is just the complementary error function which is widely tabulated and for which there are simple computer approximations.

For $z > t^*$, it is convenient to calculate J by means of $K = \int_{-\infty}^{\infty} - \int_{z}^{\infty}$.
Then an approximation of the type discussed in the previous paragraph gives

$$J \doteq I - \frac{e^{\frac{-z^2}{2} - E}}{\pi^{1/2}(1+b^2E)^{1/2}} e^{\frac{(bE+z)^2}{2(1+b^2E)}} \int_{s=\frac{bE+z}{2^{1/2}(1+b^2E)^{1/2}}}^{\infty} e^{-s^2} ds \qquad (z>t^*) \qquad \text{(A4-3)}$$

Note that for $z = t^*$ the sum of the approximations to $\int_{-\infty}^{z}$ and $\int_{z}^{\infty}$ just adds up to the approximation given for $\int_{-\infty}^{\infty}$.